# FILTRATIONS OF SYMPLECTIC VECTOR BUNDLES ON THE FARGUES-FONTAINE CURVE

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Abstract. Given a symplectic vector bundles on the Fargues-Fontaine curve, we present a sufficient condition for the existence of an isotropic filtration with specified successive quotients.

Contents



## 1. Introduction

The *Fargues-Fontaine curve* is a fundamental object in modern arithmetic geometry. While its original construction by Fargues-Fontaine [FF18] stems from $p$-adic Hodge theory, it has led to numerous breakthroughs in $p$-adic geometry and the Langlands program as pioneered by the work of Scholze-Weinstein [SW20] and Fargues-Scholze [FS21]. A common theme for these applications is to describe various objects in arithmetic geometry in terms of $G$-bundles on the Fargues-Fontaine curve for a $p$-adic group $G$.

An important problem for studying $G$-bundles on the Fargues-Fontaine curve is the classification of their parabolic reductions. In fact, this problem naturally arises in the study of local Shimura varieties, $p$-adic flag varieties, or the $B_{\mathrm{dR}}^+$-Grassmanninans. For $G = \mathrm{GL}_n$, parabolic reductions simply mean filtrations by subbundles. In this case, their classification is given by the work of the author [Hon25] and Chen-Tong [CT22]. The main objective of this article is to establish an analogous classification for $G = \mathrm{Sp}_{2n}$.

In order to state our main result, let us set up some basic notations and collect some fundamental facts about the Fargues-Fontaine curve. Let $E$ be a finite extension of $\mathbb{Q}_p$ for a fixed prime $p$ and $F$ be an algebraically closed nonarchimedean complete field of characteristic $p$. We write $X_{E,F}$ for the Fargues-Fontaine curve associated to the pair $(E, F)$, which is a noetherian adic space over $E$ with Picard group $\mathbb{Z}$. The main result of Fargues-Fontaine [FF18] shows that vector bundles on $X_{E,F}$ form a slope category where Harder-Narasimhan filtrations (noncanonically) split. Given a vector bundle $\mathcal{V}$ on $X_{E,F}$, we denote its slope by $\mu(\mathcal{V})$ and its dual by $\mathcal{V}^\vee$. In addition, we regard the Harder Narasimhan polygon $\mathrm{HN}(\mathcal{V})$ of $\mathcal{V}$ as a piecewise linear function on the interval $[0, \mathrm{rk}(\mathcal{V})]$ and write $\mu_i(\mathrm{HN}(\mathcal{V}))$ for its slope on the interval $[i-1, i]$. If $\mathcal{V}$ is endowed with a symplectic pairing, we define its *symplectic filtration* to be a filtration of the form

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{2n} = \mathcal{V}$$

such that each $\mathcal{V}_i$ for $i \leq n$ is an isotropic subbundle of $\mathcal{V}$ with symplectic complement $\mathcal{V}_{2n-i}$.

**Theorem 1.1.** *Let $\mathcal{E}$ be a symplectic vector bundle on $X_{E,F}$ of rank $2n$. Given vector bundles $\mathcal{F}_1, \cdots, \mathcal{F}_m$ on $X_{E,F}$ with $\mu(\mathcal{F}_i) \leq 0$, there exists a symplectic filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{2m} = \mathcal{E}$$

*with $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{F}_i$ for each $i = 1, \cdots, m$ if there exists a vector bundle $\mathcal{D}$ on $X_{E,F}$ with the following properties:*

*(i) $\mathcal{D}$ satisfies the inequality*

$$\mu_i(\mathrm{HN}(\mathcal{D})) \leq \mu_{n+i+1}(\mathrm{HN}(\mathcal{E})) \qquad \text{for } i = 1, \cdots, n-1.$$

*(ii) There exists a filtration*

$$0 = \mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots \subset \mathcal{D}_m = \mathcal{D}$$

*with $\mathcal{D}_i/\mathcal{D}_{i-1} \simeq \mathcal{F}_i$ for each $i = 1, \cdots, m$.*

By a previous result of the author [Hon25], we can state both properties in Theorem 1.1 purely in terms of Harder-Narasimhan polygons. We refer the readers to Theorem 3.8 for a precise statement.

Theorem 1.1 has several applications in studying the geometry of $p$-adic flag varieties via a natural stratification called the *Newton stratification*. We plan to address these applications in a subsequence manuscript.

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## 2. Extensions of vector bundles on the Fargues-Fontaine curve

Throughout this paper, we let $E$ be a finite extension of $\mathbb{Q}_p$ for a fixed prime $p$ and $F$ be a complete nonarchimedean algebraically closed field of characteristic $p$.

**Theorem 2.1** ([KL15, §8.7], [Ked16, §3])**.** *Denote the valuation rings of $E$ and $F$ respectively by $\mathcal{O}_E$ and $\mathcal{O}_F$. Choose a uniformizer $\pi$ of $E$ and a pseudouniformizer $\varpi$ of $F$. Write $q$ for the number of elements in the residue field of $E$.*

*(1) The $q$-Frobenius automorphism on $W_{\mathcal{O}_E}(\mathcal{O}_F) := W(\mathcal{O}_F) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ with Teichmuller lift $[\varpi]$ of $\varpi$ induces a properly discontinuous automorphism $\phi$ on the adic space*

$$Y := \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_F) \setminus \{|\pi[\varpi]| = 0\}.$$

*(2) The adic space $X := Y/\phi^{\mathbb{Z}}$ over $E$ is strongly noetherian and is locally the adic spectrum of a principal ideal domain.*

**Definition 2.2.** We refer to the adic space $X = Y/\phi^{\mathbb{Z}}$ in Theorem 2.1 as the *Fargues-Fontaine curve* associated to the pair $(E, F)$.

**Remark.** The Fargues-Fontaine curve has an algebraic incarnation as a scheme given by

$$X^{\text{alg}} := \text{Proj}\left(\bigoplus_{n\geq 0} H^0(Y, \mathcal{O}_Y)^{\phi=\pi^n}\right),$$

which recovers the original construction of Fargues-Fontaine [FF18]. The work of Kedlaya-Liu [KL15] establishes an analogue of GAGA for the Fargues-Fontaine curve, which in particular yields a categorical equivalence between vector bundles on $X$ and $X^{\text{alg}}$.

**Proposition 2.3** ([FF18, Proposition 8.2.6])**.** *There exists a natural exact tensor functor from the category of isocrystals over $\breve{E} := W(\overline{\mathbb{F}_q})[1/\pi]$ to the category of vector bundles on $X$.*

**Remark.** In fact, given an isocrystal $D$ over $\breve{E}$, the associated vector bundle on $X = Y/\phi^{\mathbb{Z}}$ is given by descending the $\mathcal{O}_Y$-module $D \otimes_{\breve{E}} \mathcal{O}_Y$ via $\phi_D \otimes \phi$, where $\phi_D$ denotes the Frobenius automorphism of $D$.

**Definition 2.4.** For each $\lambda \in \mathbb{Q}$, we define $\mathcal{O}(\lambda)$ to be the vector bundle on $X$ associated to the simple isocrystal over $\breve{E}$ of slope $\lambda$ under the functor given by Proposition 2.3.

**Proposition 2.5** ([FF18, Théorème 6.5.2])**.** *The Picard group* $\text{Pic}(X)$ *of $X$ admits a natural isomorphism* $\text{Pic}(X) \cong \mathbb{Z}$ *which associates each $d \in \mathbb{Z}$ to the isomorphism class of $\mathcal{O}(d)$.*

**Definition 2.6.** Let $\mathcal{V}$ be a nonzero vector bundle on $X$.

(1) We write $\text{rk}(\mathcal{V})$ for the rank of $\mathcal{V}$ and define the *degree* of $\mathcal{V}$ to be the integer $\deg(\mathcal{V})$ with $\wedge^{\text{rk}(\mathcal{V})}(\mathcal{V}) \cong \mathcal{O}(\deg(\mathcal{V}))$.

(2) We define the *slope* of $\mathcal{V}$ to be

$$\mu(\mathcal{V}) := \frac{\deg(\mathcal{V})}{\text{rk}(\mathcal{V})}.$$

**Proposition 2.7** ([FF18, Proposition 5.6.23],[Ked05, Proposition 4.1.3])**.** *Let $\lambda = d/r$ be a rational number written in lowest terms with $r > 0$.*

*(1) The vector bundle $\mathcal{O}(\lambda)$ has rank $r$, degree $d$, and slope $\lambda = d/r$.*

*(2) The dual of $\mathcal{O}(\lambda)$ is canonically isomorphic to $\mathcal{O}(-\lambda)$.*

*(3)* $\text{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\lambda'))$ *vanishes for each rational number $\lambda' < \lambda$.*

*(4)* $\text{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\lambda'))$ *vanishes for each rational number $\lambda' \geq \lambda$.*

**Definition 2.8.** Let $\mathcal{V}$ be a nonzero vector bundle on $X$.

(1) We say that $\mathcal{V}$ is *stable* if we have $\mu(\mathcal{W}) < \mu(\mathcal{V})$ for every nonzero subbundle $\mathcal{W} \neq \mathcal{V}$.

(2) We say that $\mathcal{V}$ is *semistable* if we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$ for every nonzero subbundle $\mathcal{W}$.

**Theorem 2.9** ([FF18, Théorème 8.2.10], [Ked05, Theorem 4.5.7])**.** *Let $\mathcal{V}$ be a nonzero vector bundle on $X$.*

*(1) $\mathcal{V}$ is stable of slope $\lambda$ if and only if it is isomorphic to $\mathcal{O}(\lambda)$.*

*(2) $\mathcal{V}$ is semistable of slope $\lambda$ if and only if it is isomorphic to $\mathcal{O}(\lambda)^{\oplus m}$ for some $m > 0$.*

*(3) $\mathcal{V}$ admits a direct sum decomposition*

$$\mathcal{V} \simeq \bigoplus_{i=1}^{l} \mathcal{O}(\lambda_i)^{\oplus m_i} \quad \text{with } \lambda_i \in \mathbb{Q},$$

*where the $\lambda_i$'s are all distinct and uniquely determined up to permutations.*

**Definition 2.10.** Let $\mathcal{V}$ be a nonzero vector bundle on $X$. Fix a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{l} \mathcal{O}(\lambda_i)^{\oplus m_i} \qquad \text{with } \lambda_i \in \mathbb{Q}$$

given by Theorem 2.9.

(1) We refer to the numbers $\lambda_i$ as the *Harder-Narasimhan (HN) slopes* of $\mathcal{V}$, or often simply as the *slopes* of $\mathcal{V}$, and write $\mu_{\max}(\mathcal{V})$ and $\mu_{\min}(\mathcal{V})$ respectively for the maximum and minimum HN slopes of $\mathcal{V}$.
(2) For every $\mu \in \mathbb{Q}$, we define the direct summands

$$\mathcal{V}^{\geq \mu} := \bigoplus_{\lambda_i \geq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i} \qquad \text{and} \qquad \mathcal{V}^{\leq \mu} := \bigoplus_{\lambda_i \leq \mu} \mathcal{O}(\lambda_i)^{\oplus m_i},$$

and similarly define $\mathcal{V}^{>\mu}$ and $\mathcal{V}^{<\mu}$.
(3) We define the *Harder-Narasimhan (HN) polygon* of $\mathcal{V}$, denoted by $\mathrm{HN}(\mathcal{V})$, as the upper convex hull of the points $(0, 0)$ and $\big(\mathrm{rk}(\mathcal{V}^{\geq \lambda_i}), \deg(\mathcal{V}^{\geq \lambda_i})\big)$.

**Definition 2.11.** A *rationally tuplar polygon* is a graph $\mathscr{P}$ of a continuous real-valued function on a closed interval with the following properties:

(i) The endpoints of $\mathscr{P}$ are integer points, with the left endpoint being $(0, 0)$.
(ii) For each integer $i$ in the domain, $\mathscr{P}$ is linear on the interval $[i-1, i]$ with a rational slope denoted by $\mu_i(\mathscr{P})$.

**Remark.** For every vector bundle $\mathcal{V}$ on $X$, its Harder-Narasimhan polygon $\mathrm{HN}(\mathcal{V})$ is a rationally tuplar polygon with the following additional properties:

- All breakpoints of $\mathrm{HN}(\mathcal{V})$ are integer points.
- The slopes of $\mathrm{HN}(\mathcal{V})$ are decreasing.

However, in this article we will consider rationally tuplar polygons which do not necessarily enjoy these properties.

**Definition 2.12.** Let $\mathscr{P}$ and $\mathscr{Q}$ be rational tuplar polygons. We say that $\mathscr{P}$ *dominates* $\mathscr{Q}$ and write $\mathscr{P} \geq \mathscr{Q}$ if $\mathscr{P}$ and $\mathscr{Q}$ satisfy the following properties:

(i) $\mathscr{P}$ and $\mathscr{Q}$ have the same endpoints.
(ii) If $r$ denotes the $x$-coordinate of their common right endpoint, we have

$$\sum_{i=1}^{j} \mu_i(\mathscr{P}) \geq \sum_{i=1}^{j} \mu_i(\mathscr{Q}) \qquad \text{for each } j = 1, \cdots, r.$$

**Remark.** In other words, we have $\mathscr{P} \geq \mathscr{Q}$ if and only if $\mathscr{P}$ lies on or above $\mathscr{Q}$ with the same endpoints, as in Mazur's inequality.

**Proposition 2.13** ([BFH+22, Theorem 1.1.2])**.** *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X$ such that $\mathcal{D}$ and $\mathcal{F}$ are semistable with $\mu(\mathcal{D}) \leq \mu(\mathcal{F})$. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

*if and only if we have* $\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

**Proposition 2.14** ([Hon23, Theorem 1.1.1])**.** *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X$ such that $\mathcal{E}$ is semistable. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

*if and only if we have* $\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ *and* $\mu_{max}(\mathcal{D}) \leq \mu(\mathcal{E}) \leq \mu_{min}(\mathcal{F})$.

**Definition 2.15.** Given vector bundles $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ on $X$, we define a $(\mathcal{D}, \mathcal{E}, \mathcal{F})$*-permutation* of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ to be a rationally tuplar polygon $\mathscr{P} \geq \mathrm{HN}(\mathcal{E})$ with the following properties:

(i) The tuple $(\mu_i(\mathscr{P}))$ is a permutation of the tuple $(\mu_i(\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})))$.

(ii) For each $i = 1, \cdots, \mathrm{rk}(\mathcal{E})$, we have
- $\mu_i(\mathscr{P}) < \mu_i(\mathrm{HN}(\mathcal{E}))$ only if $\mu_i(\mathscr{P})$ occurs as a slope of $\mathcal{D}$, and
- $\mu_i(\mathscr{P}) > \mu_i(\mathrm{HN}(\mathcal{E}))$ only if $\mu_i(\mathscr{P})$ occurs as a slope of $\mathcal{F}$.

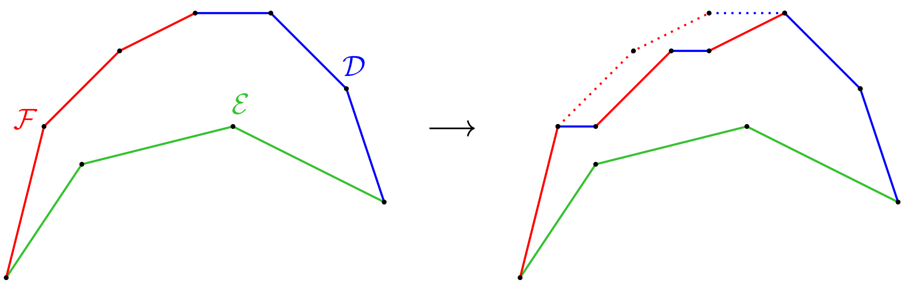


FIGURE 1. Illustration of the conditions in Definition 2.15

**Proposition 2.16** ([Hon25, Theorem 3.12])**.** *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X$ with a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

*There exists a $(\mathcal{D}, \mathcal{E}, \mathcal{F})$-permutation of* $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

**Definition 2.17.** Given a vector bundle $\mathcal{V}$ on $X$, its *semistable factor sequence* is a sequence of vector bundles $\mathcal{V}_1, \cdots, \mathcal{V}_r$ on $X$ such that $\mathrm{HN}(\mathcal{V}_i)$ represents the $i$-th line segment in $\mathrm{HN}(\mathcal{V})$.

**Proposition 2.18** ([Hon25, Theorem 4.4])**.** *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on $X$. Take a semistable factor sequence $\mathcal{F}_1, \cdots, \mathcal{F}_r$ of $\mathcal{F}$. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

*if and only if the following equivalent conditions are satisfied:*

*(i) There exists a filtration*

$$\mathcal{D} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

*with $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{F}_i$ for each $i = 1, \cdots, r$.*

*(ii) There exists a sequence of vector bundles $\mathcal{D} = \mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_r = \mathcal{E}$ such that the polygon* $\mathrm{HN}(\mathcal{E}_{i-1} \oplus \mathcal{F}_i)$ *admits an $(\mathcal{E}_{i-1}, \mathcal{E}_i, \mathcal{F}_i)$-permutation for each $i = 1, \cdots, r$.*

## 3. FILTRATIONS OF SYMPLECTIC VECTOR BUNDLES

Our goal in this section is to state and derive the necessary conditions for the existence of a short exact sequence involving three given vector bundles on the Fargues-Fontaine curve.

**Definition 3.1.** A *symplectic vector bundle* on $X$ is a vector bundle $\mathcal{E}$ on $X$ endowed with a symplectic pairing $\omega_{\mathcal{E}} : \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{O}_X$.

**Lemma 3.2.** *The HN polygon of a symplectic vector bundle $\mathcal{E}$ on $X$ is symmetric.*

*Proof.* Since the symplectic pairing of $\mathcal{E}$ induces an isomorphism between $\mathcal{E}$ and its dual $\mathcal{E}^\vee$, the assertion is evident by Proposition 2.7. $\square$

**Definition 3.3.** Let $\mathcal{E}$ be a symplectic vector bundle on $X$ and $\mathcal{D}$ be a subbundle of $\mathcal{E}$.

(1) We say that $\mathcal{D}$ is *isotropic* if it lies in its symplectic complement $\mathcal{D}^\perp$.
(2) We say that $\mathcal{D}$ is *Lagrangian* if it coincides with its symplectic complement $\mathcal{D}^\perp$.

**Lemma 3.4.** *Given a symplectic vector bundle $\mathcal{E}$ on $X$ of rank $2n$, every Lagrangian subbundle of $\mathcal{E}$ is of rank $n$.*

*Proof.* The assertion is straightforward to verify. □

**Definition 3.5.** Given a symplectic vector bundle $\mathcal{E}$ on $X$, its *symplectic filtration* is a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{2m} = \mathcal{E}$$

such that each $\mathcal{E}_i$ for $i = 1, \cdots, m$ is isotropic with $\mathcal{E}_i^\perp = \mathcal{E}^{2m-i}$.

**Remark.** If we regard $\mathcal{E}$ as an $\mathrm{Sp}_{2n}$-bundle on $X$ with $2n = \mathrm{rk}(\mathcal{E})$, a symplectic filtration of $\mathcal{E}$ corresponds to a parabolic reduction of $\mathcal{E}$.

**Proposition 3.6** ([Vie24, Theorem 1.2])**.** *Let $\mathcal{E}$ be a symplectic vector bundle on $X$. Given semistable vector bundles $\mathcal{F}_1, \cdots, \mathcal{F}_m$ on $X$ with $\mu(\mathcal{F}_m) \leq \cdots \leq \mu(\mathcal{F}_1) \leq 0$, there exists a symplectic filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{2m} = \mathcal{E}$$

*with $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{F}_i$ for each $i = 1, \cdots, m$ if and only if we have*

$$\mathrm{HN}(\mathcal{E}) \leq \mathrm{HN}(\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m \oplus \mathcal{F}_m^\vee \oplus \cdots \oplus \mathcal{F}_1^\vee).$$

**Proposition 3.7.** *Let $\mathcal{E}$ be a symplectic vector bundle on $X$ of rank $2n$ which admtis a Lagrangian subbundle $\mathcal{D}$ with $\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{D}^\vee$. Given a vector bundle $\mathcal{K}$ on $X$ of rank $n$ with*

$$\mu_i(\mathrm{HN}(\mathcal{K})) \leq \mu_{i+1}(\mathrm{HN}(\mathcal{D})) \qquad \text{for } i = 1, \cdots, n-1, \tag{3.1}$$

*there exists a Lagrange subbundle of $\mathcal{E}$ which is isomorphic to $\mathcal{K}$.*

*Proof.* Take a semistable factor sequence $\mathcal{K}_1, \cdots, \mathcal{K}_r$ of $\mathcal{K}$. If we have $r = 1$, we find

$$\mathrm{HN}(\mathcal{K} \oplus \mathcal{K}^\vee) \geq \mathrm{HN}(\mathcal{D} \oplus \mathcal{D}^\vee) = \mathrm{HN}(\mathcal{E})$$

and in turn deduce the desired assertion by Proposition 3.6. Let us henceforth assume the inequality the inequality $r > 1$ and proceed by induction on $r$. Readers may refer to Figure 2 for a visual illustration. We obtain a decomposition

$$\mathcal{K} \simeq \mathcal{K}_1 \oplus \overline{\mathcal{K}}$$

where $\overline{\mathcal{K}}$ is a vector bundle on $X$ with $r-1$ distinct slopes in $\mathrm{HN}(\overline{\mathcal{K}})$ and $\mu_{\max}(\overline{\mathcal{K}}) < \mu(\mathcal{K}_1)$. Moreover, we apply the inequality (3.1) to find

$$\mathrm{rk}(\mathcal{K}_1) \leq \mathrm{rk}(\mathcal{D}^{\geq \mu(\mathcal{K}_1)}) - 1. \tag{3.2}$$

Take $\mathcal{Q}$ be the semistable vector bundle on $X$ with

$$\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\mathcal{D}^{\geq \mu(\mathcal{K}_1)}) - \mathrm{rk}(\mathcal{K}_1) \quad \text{and} \quad \deg(\mathcal{Q}) = \deg(\mathcal{D}^{\geq \mu(\mathcal{K}_1)}) - \deg(\mathcal{K}_1).$$

We obtain the relation $\mathrm{HN}(\mathcal{D}^{\geq \mu(\mathcal{K}_1)}) \leq \mathrm{HN}(\mathcal{K}_1 \oplus \mathcal{Q})$, which implies by Proposition 2.13 that there exists a short exact sequence

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{D}^{\geq \mu(\mathcal{K}_1)} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Since we have $\mathcal{D} \simeq \mathcal{D}^{\geq \mu(\mathcal{K}_1)} \oplus \mathcal{D}^{< \mu(\mathcal{K}_1)}$, we identify $\mathcal{K}_1$ as a subbundle of $\mathcal{D}$ with

$$\mathcal{D}/\mathcal{K}_1 \simeq \mathcal{Q} \oplus \mathcal{D}^{< \mu(\mathcal{K}_1)}.$$

and use the inequality (3.1) to find

$$\mu_i(\mathrm{HN}(\overline{\mathcal{K}})) \leq \mu_{i+1}(\mathcal{Q} \oplus \mathrm{HN}(\mathcal{D}^{<\mu(\mathcal{K}_1)})) = \mu_{i+1}(\mathcal{D}/\mathcal{K}_1) \quad \text{for } i = 1, \cdots, n - \mathrm{rk}(\mathcal{K}_1) - 1.$$

Moreover, it is not hard to see that the symplectic vector bundle $\mathcal{K}_1^\perp/\mathcal{K}_1$ admits a Lagrangian subbundle $\mathcal{D}/\mathcal{K}_1$ with $\mathcal{K}_1^\perp/\mathcal{K}_1 \simeq \mathcal{D}/\mathcal{K}_1 \oplus (\mathcal{D}/\mathcal{K}_1)^\vee$. Therefore, by the induction hypothesis, there exists a Lagrangian subbundle of $\mathcal{K}_1^\perp/\mathcal{K}_1$ which is isomorphic to $\overline{\mathcal{K}}$. Now we establish the desired assertion as every extension of $\overline{\mathcal{K}}$ by $\mathcal{K}_1$ splits by Proposition 2.7. □

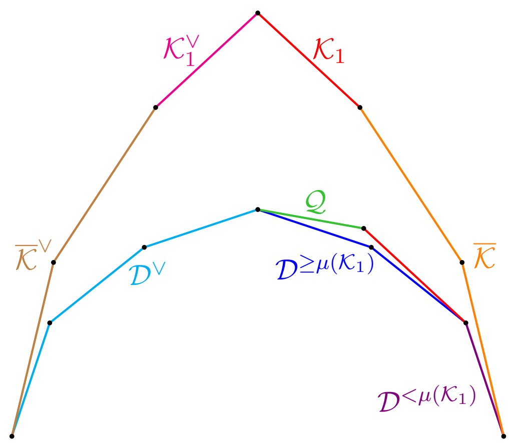


FIGURE 2. Induction step for Proposition 3.7

**Theorem 3.8.** *Let $\mathcal{E}$ be a symplectic vector bundle on $X$ of rank $2n$. Given vector bundles $\mathcal{F}_1, \cdots, \mathcal{F}_m$ on $X$ with $\mu(\mathcal{F}_i) \leq 0$, there exists a symplectic filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{2m} = \mathcal{E} \tag{3.3}$$

*with $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{F}_i$ for each $i = 1, \cdots, m$ if there exists a vector bundle $\mathcal{D}$ on $X$ of rank $n$ with the following properties:*

*(i) $\mathcal{D}$ satisfies the inequality*

$$\mu_i(\mathrm{HN}(\mathcal{D})) \leq \mu_{n+i+1}(\mathrm{HN}(\mathcal{E})) \qquad \text{for } i = 1, \cdots, n-1.$$

*(ii) For a sequence $\mathcal{Q}_1, \cdots, \mathcal{Q}_l$ given by concatenating semistable factor sequences of $\mathcal{F}_1, \cdots, \mathcal{F}_m$, there exists a sequence of vector bundles $0 = \mathcal{K}_0, \mathcal{K}_1, \cdots, \mathcal{K}_l = \mathcal{D}$ on $X$ such that $\mathrm{HN}(\mathcal{K}_{j-1} \oplus \mathcal{Q}_j)$ admits a $(\mathcal{K}_{j-1}, \mathcal{K}_j, \mathcal{Q}_j)$-permutation for each $j = 1, \cdots, l$.*

*Proof.* It is not hard to see that $\mathcal{E}$ admits a Lagrangian subbundle $\mathcal{E}^-$ with

$$\mu_i(\mathrm{HN}(\mathcal{E}^-)) = \mu_{n+i}(\mathrm{HN}(\mathcal{E})) \qquad \text{for } i = 1, \cdots, n,$$

for example by Proposition 3.6. Hence Proposition 3.7 and property (i) together imply that there exists a Lagrangian subbundle of $\mathcal{E}$ isomorphic to $\mathcal{D}$. In addition, Proposition 2.18 and property (ii) together yield a filtration

$$0 = \mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots \subset \mathcal{D}_m = \mathcal{D}$$

with $\mathcal{D}_i/\mathcal{D}_{i-1} \simeq \mathcal{F}_i$ for each $i = 1, \cdots, m$. Now we obtain the desired filtration (3.4) with $\mathcal{E}_i = \mathcal{D}_i$ and $\mathcal{E}_{2m-i} = \mathcal{D}_i^\perp$ for each $i = 1, \cdots, m$. □

**Remark.** We can extend the remark following [Hon25, Theorem 4.4] to note that the property (ii) can be checked in a finite amount of time.

**Proposition 3.9.** *Let $\mathcal{E}$ be a symplectic vector bundle on $X$. Given vector bundles $\mathcal{F}_1, \cdots, \mathcal{F}_m$ on $X$ with $\mu(\mathcal{F}_i) \leq 0$, there exists a symplectic filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{2m} = \mathcal{E} \tag{3.4}$$

*with $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{F}_i$ for each $i = 1, \cdots, m$ if the following conditions are satisfied:*

*(i) $\mathcal{E}$ satisfies the relations*

$$\mathrm{rk}(\mathcal{E}) = 2(\mathrm{rk}(\mathcal{F}_1) + \cdots + \mathrm{rk}(\mathcal{F}_m)) \quad \text{and} \quad \deg(\mathcal{E}^{\leq 0}) \geq \deg(\mathcal{F}_1) + \cdots + \deg(\mathcal{F}_m).$$

*(ii) $\mathcal{F}_1$ satisfies the inequality*

$$\mu_{min}(\mathcal{F}_1) \geq \frac{\deg(\mathcal{F}_1) + \deg(\mathcal{F}_2)}{\mathrm{rk}(\mathcal{F}_1) + \mathrm{rk}(\mathcal{F}_2)},$$

*while each $\mathcal{F}_i$ with $i > 1$ satisfies the inequality*

$$\mu_{max}(\mathcal{F}_i) \leq \frac{\deg(\mathcal{F}_1) + \cdots + \deg(\mathcal{F}_i)}{\mathrm{rk}(\mathcal{F}_1) + \cdots + \mathrm{rk}(\mathcal{F}_i)}.$$

*Proof.* Set $\mathcal{E}_1 = \mathcal{F}_1$ and take $\mathcal{E}_i$ for each $i = 2, \cdots, m$ to be a semistable vector bundle on $X$ with

$$\mathrm{rk}(\mathcal{E}_i) = \mathrm{rk}(\mathcal{F}_1) + \cdots + \mathrm{rk}(\mathcal{F}_i) \quad \text{and} \quad \deg(\mathcal{E}_i) = \deg(\mathcal{F}_1) + \cdots + \deg(\mathcal{F}_i).$$

It is not hard to see by Proposition 3.6 and the condition (i) that $\mathcal{E}$ admits a Lagrangian subbundle isomorphic to $\mathcal{E}_m$. In addition, Proposition 2.14 and condition (ii) together yield a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m.$$

Now we obtain the desired assertion by taking $\mathcal{E}_{2m-i} = \mathcal{E}_i^{\perp}$ for each $i = 1, \cdots, m$. □

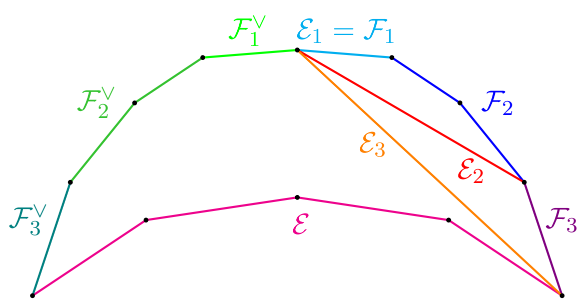


FIGURE 3. Construction of the filtration in Proposition 3.9

**Remark.** We can also establish Proposition 3.9 as a special case of Theorem 3.8.

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